

THE SCATTERING RELATION ON ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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ABSTRACT. We study the scattering relation and the sojourn times on non-trapping asymptotically hyperbolic manifolds and use it to obtain the asymptotics of the distance function on geodesically convex asymptotically hyperbolic manifolds.

1. INTRODUCTION

Asymptotically hyperbolic manifolds are a generalization of the hyperbolic space (\mathbb{B}^{n+1}, g) ,

$$(1.1) \quad \mathbb{B}^{n+1} = \{z \in \mathbb{R}^{n+1} : |z| < 1\} \text{ equipped with the metric } g = \frac{4dz^2}{(1 - |z|^2)^2}.$$

Instead of \mathbb{B}^{n+1} , we consider the interior of a C^∞ manifold with boundary X of dimension $n + 1$, and assume that the interior of X , which we denote by \mathring{X} , is equipped with a metric g such that for any defining function ρ of ∂X (i.e. $\rho \in C^\infty(X)$, $\rho > 0$ in \mathring{X} , $\{\rho = 0\} = \partial X$ and $d\rho \neq 0$ at ∂X), $\rho^2 g$ is a C^∞ non-degenerate Riemannian metric up to ∂X . In the case of the hyperbolic space $X = \mathbb{B}^{n+1}$ and $\rho = 1 - |z|^2$. According to [8] the manifold (\mathring{X}, g) is complete and its sectional curvatures approach $-|d\rho|_{\partial X}|_{h_0}^2$, as $\rho \downarrow 0$ along any curve, where $h_0 = \rho^2 g|_{\partial X}$. In particular, when

$$(1.2) \quad |d\rho|_{\partial X}|_{h_0} = 1,$$

the sectional curvature converges to -1 at the boundary. Following Mazzeo and Melrose [8], manifolds (\mathring{X}, g) for which $\rho^2 g$ is non-degenerate at ∂X and (1.2) holds are called asymptotically hyperbolic manifolds (AHM). It follows from the definition that the metric g determines a conformal structure on ∂X , and because of that these manifolds have been studied in connection with conformal field theory [3, 4]. As shown in [3, 7], if $h_0 \in [\rho^2 g|_{\partial X}]$, the equivalence class of $\rho^2 g|_{\partial X}$, there exists a boundary defining function x in a neighborhood of ∂X such that

$$(1.3) \quad g = \frac{dx^2}{x^2} + \frac{h(x)}{x^2}, \quad h(0) = h_0, \text{ on } [0, \varepsilon) \times \partial X,$$

where $h(x)$ is a C^∞ family of Riemannian metrics on ∂X parametrized by x . Of course, x can be extended (non-uniquely) to X by setting it equal to a constant on a compact set of \mathring{X} .

We say that (\mathring{X}, g) is non-trapping if any geodesic $\gamma(t) \rightarrow \partial X$ as $\pm t \rightarrow \infty$, and we shall assume throughout this paper that (\mathring{X}, g) is non-trapping. Our goal is to understand the behavior of geodesics on non-trapping AHM and define the scattering relation at the boundary at infinity. One can easily describe the scattering relation for non-trapping compactly supported metric perturbations of Euclidean space. Suppose that $g = \sum_{i,j=1}^n g_{ij}(x) dx_i dx_j$ is a C^∞ non-trapping Riemannian metric on \mathbb{R}^n and suppose that $g_{ij}(x) = \delta_{ij}$ if $x \notin K \subset \mathbb{R}^n$, where K is compact. Let B be a bounded ball and suppose $K \subset B$. A light ray comes from $\mathbb{R}^n \setminus B$ enters B , is scattered

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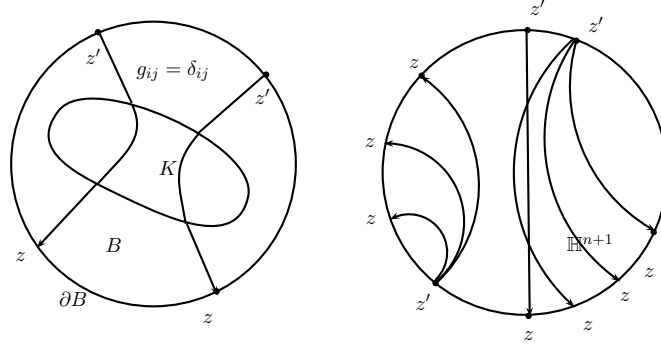


FIGURE 1. The scattering relation for compactly supported perturbations of the Euclidean metric and for hyperbolic space.

by the metric in K and comes out of B . If the light ray that comes into B intersects ∂B at a point $z \in \partial B$ in the direction ζ and comes out at points $z' \in \partial B$ with direction ζ' , the map $(z, \zeta) \mapsto (z', \zeta')$ is called the scattering relation. The time t that it takes for the geodesic to travel across B is called the sojourn time. This can also be described in terms of the submanifold $\Lambda = \{(t, 1, z, \zeta, z', \zeta') : (z, \zeta) = \exp(tH_q)(z', \zeta')\}$, where $q = \sum_{i,j=1}^n g^{ij} \xi_i \xi_j$, $g^{-1} = (g^{ij})$ is the dual metric to g . The scattering relation is then the restriction of Λ to $\partial B \times \partial B$, i.e. $\Lambda \cap (T_{\partial B}^* \mathbb{R}^n \times T_{\partial B}^* \mathbb{R}^n)$. Theorem 1.1 below can be used to define the analogue of such a map for non-trapping AHM. Let Λ_1 be the extension of $\beta_1^* \tilde{\Lambda}$ up to $\partial(\mathbb{R}_s \times X \times_0 X)$, as defined below. The scattering relation is $\Lambda_1 \cap T_{\mathbb{R}_s \times \{\rho_R = \rho_L = 0\}}^*(\mathbb{R}_s \times X \times_0 X)$.

There is a long list of papers dedicated to the scattering relation in different settings, and it would be impossible to give a precise history. Guillemin [5] studied the scattering relation and sojourn times for scattering by a convex obstacle and scattering for the automorphic wave equation after Faddeev-Lax-Phillips. Uhlmann [17] showed that the Dirichlet-to-Neumann map for the wave equation gives the scattering relation on a manifold with boundary without any assumptions on caustics, while a similar result had been proved by Sylvester and Uhlmann [16] when there are no conjugate points. Melrose, Sá Barreto and Vasy [10] studied the scattering relation for certain perturbations of the hyperbolic space. The scattering relation has also been studied in locally symmetric spaces, see for example the work of Ji and Zworski [6] and references cited there. Uhlmann, Pestov and Uhlmann, Stefanov and Uhlmann [11, 13, 14, 15] studied the lens rigidity and boundary rigidity inverse problems, where one wants to obtain information about the manifold from the scattering relation.

Let $z, z' \in \mathring{X}$, and let γ be a geodesic joining z and z' , and let $z \rightarrow y \in \partial X$. Sá Barreto and Wunsch proved in [12] that if ρ is a defining function of ∂X , and \mathring{X} is non-trapping, the limit $s_\gamma(z', y) = \lim_{t \rightarrow \infty} (t + \log \rho(\gamma(t)))$ exists and moreover $s_\gamma(z', y)$ is a C^∞ function of z', y and the co-vector $\zeta \in T_z^* \mathring{X}$ that defines the geodesic γ . Of course, $s_\gamma(z', y)$ depends on the choice of ρ . One of our goals here is to generalize this result to the case where both points z and z' joined by a geodesic γ are allowed to go to ∂X , see Fig.2.

The metric g on $T\mathring{X}$ induces a dual metric g^* on $T^*\mathring{X}$ defined in (2.2). We shall view a geodesic as the projection of an integral curve of H_p , the Hamilton vector field of $p(z, \zeta) = \frac{1}{2}(|\zeta|_{g^*(z)} - 1)$, see Section 2. The integral curve of H_p connects two points (z', ζ') and (z, ζ) , and to better understand the map $(z', \zeta') \mapsto (z, \zeta)$, it will be convenient to work on the product $\mathring{X} \times \mathring{X}$. We can identify $T^*(\mathring{X} \times \mathring{X}) = T^*\mathring{X} \times T^*\mathring{X}$ and according to this, we shall use (z, ζ, z', ζ') to denote a point in

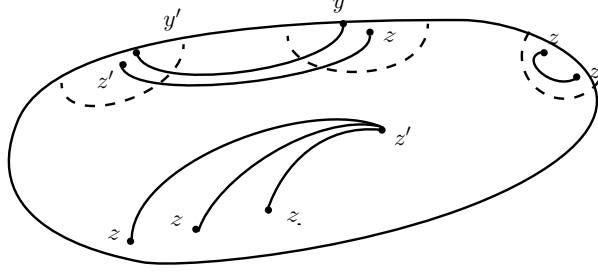


FIGURE 2. Different scenarios of points approaching ∂X : either z' is fixed and $z \rightarrow \partial X$ or $z, z' \rightarrow \partial X$, but z and z' are far apart, or $z, z' \rightarrow \partial X$ and the points are close.

$T^*(\mathring{X} \times \mathring{X})$, while (z, ζ) will denote a point on the left factor and (z', ζ') will denote a point on the right factor. In fact we shall work on $T^*(\mathbb{R} \times \mathring{X} \times \mathring{X})$, and we denote

$$Q_R(\tau, z, \zeta, z', \zeta') = \frac{1}{2}(|\zeta'|_{g^*(z')} - \tau^2) \text{ and } Q_L(\tau, z, \zeta, z', \zeta') = \frac{1}{2}(|\zeta|_{g^*(z)} - \tau^2).$$

In Section 2, we study the Lagrangian submanifold $\tilde{\Lambda} \subset T^*(\mathbb{R}_t \times \mathring{X} \times \mathring{X})$ defined as the flow out of

$$(1.4) \quad \{(0, \tau, z, \zeta, z', \zeta') : z = z', \zeta = -\zeta', \tau^2 = |\zeta|_{g^*(z)}\},$$

under the Hamilton vector field H_{Q_R} or H_{Q_L} . Since \mathring{X} is non-trapping, $\tilde{\Lambda}$ is a C^∞ Lagrangian submanifold of $T^*(\mathbb{R}_t \times \mathring{X} \times \mathring{X})$, and we will analyze the global behavior of $\tilde{\Lambda}$ up to $\partial(\mathbb{R}_t \times X \times X)$. The obvious problem is when the closure of $\text{Diag} = \{(z, z') : z = z', z \in \mathring{X}\}$ meets $\partial X \times \partial X$, where the Hamilton flow is not well-defined. To handle this situation, we work in the 0-blow-up of $X \times X$ defined in Mazzeo and Melrose [8], and we recall their construction. Let

$$\partial \overline{\text{Diag}} = \{(z, z) \in \partial X \times \partial X\} = \overline{\text{Diag}} \cap (\partial X \times \partial X).$$

As a set, the 0-blown-up space is

$$X \times_0 X = (X \times X) \setminus \partial \overline{\text{Diag}} \sqcup S_{++}(\partial \overline{\text{Diag}}),$$

where $S_{++}(\partial \overline{\text{Diag}})$ denotes the inward pointing spherical bundle of $T^*_{\partial \overline{\text{Diag}}}(X \times X)$. Let

$$\beta_0 : X \times_0 X \rightarrow X \times X$$

be the blow-down map. Then $X \times_0 X$ is equipped with the topology and smooth structure of a manifold with corners such that β_0 is smooth. The manifold $X \times_0 X$ has three boundary hypersurfaces: the left and right faces $L = \beta_0^{-1}(\partial X \times \mathring{X})$, $R = \beta_0^{-1}(\mathring{X} \times \partial X)$, and the front face $\text{ff} = \beta_0^{-1}(\partial \overline{\text{Diag}})$. The lifted diagonal is denoted by $\text{Diag}_0 = \beta_0^{-1}(\text{Diag})$. See Figure 3.

In the interior of $X \times_0 X$, β_0 is a diffeomorphism between open C^∞ manifolds, and $\beta_0^* \tilde{\Lambda}$ is naturally well-defined as the joint flow-out of the lift of (1.4) under the lifts $\beta_0^* H_{Q_R}$ and $\beta_0^* H_{Q_L}$. By abuse of notation, we will also denote

$$\begin{aligned} \beta_0 : \mathbb{R}_t \times X \times_0 X &\longrightarrow \mathbb{R}_t \times X \times X \\ (t, m) &\longmapsto (t, \beta_0(m)). \end{aligned}$$

We will prove the following result in Section 2:

Theorem 1.1. *Let (\mathring{X}, g) be a non-trapping AHM. Let ρ_L, ρ_R be boundary defining functions of L and R respectively. Let*

$$(1.5) \quad \begin{aligned} \mathcal{M} : \mathbb{R} \times (X \times_0 X \setminus (R \cup L)) &\longrightarrow \mathbb{R} \times X \times_0 X \\ (t, m) &\longmapsto (t + \log \rho_L(m) + \log \rho_R(m), m) = (s, m) \end{aligned}$$

and define $\beta_1 = \beta_0 \circ \mathcal{M} : \mathbb{R}_s \times (X \times_0 X \setminus (R \cup L)) \longrightarrow \mathbb{R}_t \times X \times X$. Let $\tilde{\Lambda} \subset T^(\mathbb{R}_t \times \mathring{X} \times \mathring{X})$ be the C^∞ Lagrangian submanifold defined in (2.5). Let $\beta_1^* \tilde{\Lambda}$ denote the lift of $\tilde{\Lambda}$ by β_1 in the interior of $X \times_0 X$. Then $\beta_1^* \tilde{\Lambda}$ has a smooth extension up to the boundary of $T^*(\mathbb{R}_s \times X \times_0 X)$.*

The main point in the proof of Theorem 1.1 is that Diag_0 does not intersect R or L and intersects ff transversally, see Fig. 3. We will also show that after the singular change of variables (1.5), $H_{\frac{1}{\rho_R}} \beta_1^* Q_R$ and $H_{\frac{1}{\rho_L}} \beta_1^* Q_L$ lift to C^∞ vector fields which are tangent to $\mathbb{R}_s \times \text{ff}$. Moreover if σ is the dual variable to s , in the region $\sigma \neq 0$, where $\beta_1^* \tilde{\Lambda}$ is contained, $H_{\frac{1}{\rho_R}} \beta_1^* Q_R$ is transversal to $\mathbb{R}_s \times R$, and $H_{\frac{1}{\rho_L}} \beta_1^* Q_L$ is transversal to $\mathbb{R}_s \times L$. Therefore, the manifold $\beta^* \tilde{\Lambda}$ extends smoothly up to $\partial(\mathbb{R}_s \times X \times_0 X)$, see Fig. 4. The main point is that after the singular change of variables and rescaling the lift of the symbols, the Lagrangian results from the integration of a C^∞ vector field over a finite interval. This idea is reminiscent from the work of Sá Barreto and Wunsch [12] and Melrose, Sá Barreto and Vasy [10].

Theorem 1.1 generalizes a result of [12] and shows that one can define the sojourn time along a geodesic joining two points z and z' , as both points go to ∂X . This will be discussed in details in Section 2.

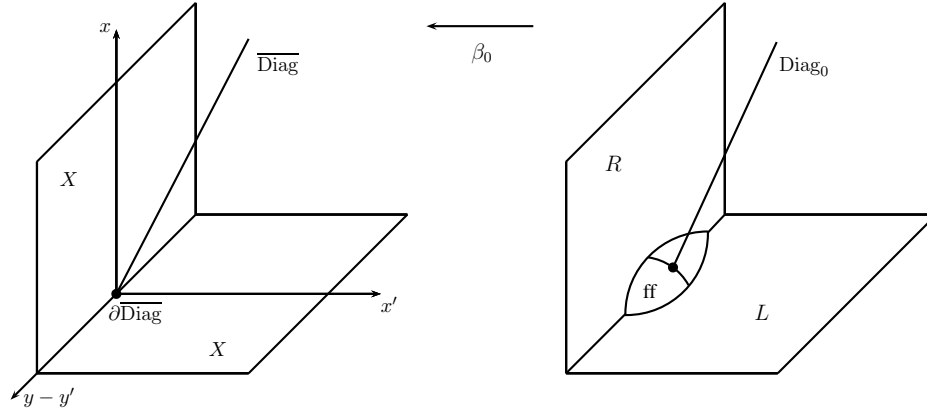


FIGURE 3. The 0-blown-up space $X \times_0 X$.

As an application of Theorem 1.1, we study the asymptotics of the distance function $r(z, z')$ between $z, z' \in \mathring{X}$ as $z, z' \rightarrow \partial X$, in the case where (\mathring{X}, g) is a geodesically convex AHM. In this case there are no conjugate points along any geodesic in \mathring{X} and $r(z, z')$ is equal to the length of the unique geodesic joining the two points. Moreover, $r(z, z')$ is smooth on $(\mathring{X} \times \mathring{X}) \setminus \text{Diag}$. This is the case when (\mathring{X}, g) is a Cartan-Hadamard manifold, i.e. when \mathring{X} has non-positive sectional curvature, see [2]. We will show in Section 3 that the following is a consequence of Theorem 1.1

Theorem 1.2. *Let (\mathring{X}, g) be a geodesically convex AHM, and let ρ_L and ρ_R be boundary defining functions of L and R respectively. For $z, z' \in \mathring{X}$, the lift of the distance function $r(z, z')$ to $X \times_0 X$*

satisfies

$$(1.6) \quad \beta_0^* r = -\log \rho_L - \log \rho_R + F, \quad F \in C^\infty(X \times_0 X \setminus \text{Diag}_0).$$

One should emphasize that the importance of (1.6) is what it reveals about the behavior of $\beta_0^* r$ near the right, left and front faces, and where Diag_0 meets ff . The singularity of $\beta_0^* r$ at the diagonal in the interior is well understood. In fact, since $r(z, z')$ is the distance function, then in the interior of $X \times_0 X$ and near Diag_0 , $(\beta_0^* r(z, z'))^2$ is C^∞ and vanishes quadratically at Diag_0 . One can always modify ρ_R and ρ_L such that $\rho_R = \rho_L = 1$ near Diag_0 , and with this choice of ρ_R and ρ_L , one has that $F = \beta_0^* r$ near Diag_0 . This regularity near Diag_0 extends up to ff .

The asymptotics of $r(z, z')$ for perturbations of the hyperbolic space of the type

$$(1.7) \quad \mathring{X} = \mathbb{B}^{n+1} \text{ equipped with the metric } g_\varepsilon = \frac{4dz^2}{(1-|z|^2)^2} + \chi\left(\frac{1-|z|^2}{\varepsilon}\right)H(z, dz),$$

where $\chi(t) \in C_0^\infty(\mathbb{R})$, with $\chi(t) = 1$ if $|t| < 1$ and $\chi(t) = 0$ if $|t| > 2$, H is a C^∞ symmetric 2-tensor and ε is small enough, was studied by Melrose, Sá Barreto and Vasy in [10] in connection with the analysis of the asymptotic behavior of solutions of the wave equation on de Sitter-Schwarzschild space-time. For ε small enough, $(\mathbb{B}^{n+1}, g_\varepsilon)$ is an example of a Cartan-Hadamard manifold. It was proved in [10] that there exists $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, then Theorem 1.2 holds for the particular case when g is given by (1.7). This was proved by first analyzing the case of the hyperbolic space and using perturbation arguments.

Equation (1.6) was the key ingredient in the construction of a semiclassical parametrix for the resolvent of the Laplacian $R(\lambda, h) = \left(h^2(\Delta_g - \frac{n^2}{4}) - \lambda^2\right)^{-1}$, when (\mathring{X}, g) satisfied (1.7) carried out in [10]. In particular, in view of (1.6), the proof of Theorem 5.1 of [10] extends line by line to geodesically convex AHM.

It is easy to illustrate Theorem 1.2 in half-space model of hyperbolic space. A similar computation is done in [10]. In this case,

$$\mathbb{H}^{n+1} = \{(x, y) : x > 0, y \in \mathbb{R}^n\}, \quad g = \frac{dx^2}{x^2} + \frac{dy^2}{x^2},$$

and the distance function satisfies

$$\cosh r(z, z') = \frac{x^2 + x'^2 + |y - y'|^2}{2xx'}, \quad z = (x, y), \quad z' = (x', y').$$

Since

$$(1.8) \quad e^r = \cosh r + \sqrt{\cosh^2 r - 1},$$

we obtain

$$r(z, z') = -\log(xx') + \log \frac{1}{2} \left(x^2 + x'^2 + |y - y'|^2 + \left[((x + x')^2 + |y - y'|^2)((x - x')^2 + |y - y'|^2) \right]^{\frac{1}{2}} \right).$$

This shows that away from the diagonal, i.e. if $(x - x')^2 + |y - y'|^2 > \delta > 0$,

$$r(z, z') = -\log(xx') + F,$$

where F is smooth up to $x = 0$ and $x' = 0$. However the asymptotic behavior of $r(z, z')$ near the diagonal as $x \downarrow 0$ and $x' \downarrow 0$ is more appropriately expressed in terms of polar coordinates. In this case, we choose $R = [x^2 + x'^2 + |y - y'|^2]^{\frac{1}{2}}$ as a defining function of the submanifold

$$\partial \overline{\text{Diag}} = \{x = 0, x' = 0, y = y'\},$$

and denote $\rho_R = x'/R$, $\rho_L = x/R$ and $Y = \frac{y-y'}{R}$. Then by using (1.8), we have

$$(1.9) \quad \beta_0^* r + \log(\rho_R \rho_L) = \log \frac{1}{2} \left(1 + [((\rho_R + \rho_L)^2 + |Y|^2)((\rho_R - \rho_L)^2 + |Y|^2)]^{\frac{1}{2}} \right),$$

which confirms (1.6) in this example.

Notice that the right hand side of (1.9) does not depend on R . So the asymptotics holds up to the front face ff . Notice also that, away from the front face

$$r(z, z') + \log(xx')|_{\{x=x'=0\}} = 2 \log |y - y'|.$$

This gives the sojourn time between two points that approach ∂X away from each other, see Fig.2. Similarly,

$$(1.10) \quad \beta_0^*(r + \log x + \log x')|_{\{\rho_R=\rho_L=0\}} = 2 \log(R|Y|)|_{\{\rho_R=\rho_L=0\}} = 2 \log R|_{\{\rho_R=\rho_L=0\}},$$

since $|Y| = 1$ when $\rho_L = \rho_R = 0$. It is important to point out the difference between the two asymptotic expansions. From (1.9) we have

$$(\beta_0^* r + \log \rho_R + \log \rho_L)|_{\{\rho_R=\rho_L=0\}} = 0.$$

According to [12], the sojourn time between points on $X \times_0 X$ is given by (1.10). Notice that, according to [7] this is precisely the singularity of the Schwartz kernel of the scattering matrix.

2. THE LAGRANGIAN MANIFOLD, THE SCATTERING RELATION AND SOJOURN TIMES

We recall some basic facts about Riemannian and symplectic geometry. The Riemannian metric g in the interior \mathring{X} induces an isomorphism

$$(2.1) \quad \begin{aligned} \mathcal{G} : T_z \mathring{X} &\longrightarrow T_z^* \mathring{X} \\ v &\longmapsto g(z)(v, \cdot), \end{aligned}$$

which in turn induces a dual metric on $T^* \mathring{X}$ given by

$$(2.2) \quad g(z)^*(\xi, \eta) = g(z)(\mathcal{G}^{-1}\xi, \mathcal{G}^{-1}\eta).$$

In local coordinates we have

$$g(z)(v, w) = \sum_{i,j} g_{ij}(z) v_i w_j \text{ and } g(z)^*(\xi, \eta) = \sum_{i,j} g^{ij}(z) \xi_i \eta_j, \text{ where the matrices } (g_{ij})^{-1} = (g^{ij}).$$

Consider the product manifold $T^*(\mathbb{R}_t \times \mathring{X} \times \mathring{X})$ which can be identified with $T^*\mathbb{R} \times T^*\mathring{X} \times T^*\mathring{X}$. In local coordinates (z, ζ, z', ζ') on $T^*(\mathring{X} \times \mathring{X})$, the canonical 2-form is

$$\tilde{\omega} = d\tau \wedge dt + \sum_{j=1}^{n+1} d\zeta_j \wedge dz_j + \sum_{j=1}^{n+1} d\zeta'_j \wedge dz'_j.$$

We shall denote the conormal bundle of the diagonal by

$$N^*\text{Diag} \setminus 0 = \{(z, \zeta, z', \zeta') : z' = z \in \mathring{X}, \zeta = -\zeta' \neq 0\}.$$

We distinguish between the lift of the wave equation associated to the right or left factor. Let

$$\square_{g_R} = \frac{1}{2}(D_t^2 - \Delta_{g_R}) \text{ and } \square_{g_L} = \frac{1}{2}(D_t^2 - \Delta_{g_L}).$$

The principal symbols of these operators are

$$(2.3) \quad \begin{aligned} Q_R(z', \zeta', t, \tau) &= \frac{1}{2}(\tau^2 - |\zeta'|_{g^*(z')}^2), \text{ and} \\ Q_L(z, \zeta, t, \tau) &= \frac{1}{2}(\tau^2 - |\zeta|_{g^*(z)}^2). \end{aligned}$$

In what follows we will think of these as functions on $T^*(\mathbb{R} \times \mathring{X} \times \mathring{X})$, and H_{Q_R} and H_{Q_L} will denote the Hamilton vector fields of Q_R and Q_L with respect to $\tilde{\omega}$. In local coordinates

$$H_{Q_\bullet} = \tau \frac{\partial}{\partial t} + \sum_{j=1}^{n+1} \frac{\partial Q_\bullet}{\partial \zeta_j} \frac{\partial}{\partial z_j} - \frac{\partial Q_\bullet}{\partial z_j} \frac{\partial}{\partial \zeta_j}, \text{ where } \bullet = R, L.$$

These vector fields obviously commute, and hence for $t_1 \geq 0$ and $t_2 \geq 0$, and $(t, \tau, z, \zeta, z', \zeta') \in T^*(\mathbb{R} \times \mathring{X} \times \mathring{X})$,

$$\exp t_2 H_{Q_L} \circ \exp t_1 H_{Q_R}(t, \tau, z, \zeta, z', \zeta') = \exp t_2 H_{Q_R} \circ \exp t_1 H_{Q_L}(t, \tau, z, \zeta, z', \zeta').$$

With this identification, if

$$(2.4) \quad \Sigma \doteq \{t = 0, z = z' \in \mathring{X}, \zeta = -\zeta' \neq 0, \tau^2 = |\zeta|_{g^*(z)}^2\},$$

we define

$$(2.5) \quad \tilde{\Lambda} \doteq \bigcup_{t \geq 0} \exp t H_{Q_R}(\Sigma) = \bigcup_{t \geq 0} \exp t H_{Q_L}(\Sigma) = \bigcup_{t_1 \geq 0, t_2 \geq 0} \exp t_2 H_{Q_L} \circ \exp t_1 H_{Q_R}(\Sigma).$$

To see the last equality, we just have to realize that if $(t, \tau, z, \zeta) = \exp(s H_{Q_R})(t_1, \tau, z', \zeta')$, then $(t_1, \tau, z', \zeta') = \exp(s H_{Q_L})(t, \tau, z, \zeta)$.

Now we will carry out the proof of Theorem 1.1. First, notice that the result is independent of the choice of ρ_R or ρ_L . If $\tilde{\rho}_L, \tilde{\rho}_R$ are boundary defining functions of the left and right faces, then $\rho_L = \tilde{\rho}_L f_L$ and $\rho_R = \tilde{\rho}_R f_R$ for some $f_L, f_R \in C^\infty(X \times_0 X)$ with $f_L > 0, f_R > 0$. If $\tilde{s} = t + \log \tilde{\rho}_L + \log \tilde{\rho}_R$ and $s = t + \log \rho_R + \log \rho_L$, then $\tilde{s} = s + \log(f_L f_R)$, and the map $(s, m) \mapsto (\tilde{s}, m)$ is a global diffeomorphism of $\mathbb{R}_s \times X \times_0 X$.

The main ingredient in the proof of Theorem 1.1 is the following

Lemma 2.1. *Let $\rho_R, \rho_L \in C^\infty(X \times_0 X)$ be defining functions of R and L respectively. Let β_1 be the map defined in (1.5) and let $q_R = -\frac{1}{\rho_R} \beta_1^* Q_R$ and $q_L = -\frac{1}{\rho_L} \beta_1^* Q_L$. Then q_R and q_L extend to functions in $C^\infty(T^*(\mathbb{R}_s \times X \times_0 X))$ and the Hamilton vector fields H_{q_L} and H_{q_R} are tangent to $\mathbb{R}_s \times \text{ff}$. Moreover, if σ is the dual variable to s , then away from $\sigma = 0$, H_{q_R} is transversal to $\mathbb{R}_s \times R$, and H_{q_L} is transversal to $\mathbb{R}_s \times L$.*

Proof. We will prove this Lemma in local coordinates valid near $\partial(\mathbb{R}_s \times X \times_0 X)$. First, we choose local coordinates $z = (x, y)$ and $z' = (x', y')$ in which (1.3) holds. Then we pick the following defining functions of ff , R and L :

$$(2.6) \quad \begin{aligned} \rho_{\text{ff}} &= [x^2 + (x')^2 + |y - y'|^2]^{\frac{1}{2}} \text{ is a defining function of } \text{ff} \\ \rho_R &= \frac{x'}{\rho_{\text{ff}}} \text{ is a defining function of the right face } R \\ \rho_L &= \frac{x}{\rho_{\text{ff}}} \text{ is a defining function of the left face } L. \end{aligned}$$

We will divide the boundary of $\mathbb{R}_s \times X \times_0 X$ in four regions:

Region 1: Near $\mathbb{R}_s \times L$, and away from $\mathbb{R}_s \times (R \cup \text{ff})$, or near $\mathbb{R}_s \times R$ and away from $\mathbb{R}_s \times (L \cup \text{ff})$.

Region 2: Near $\mathbb{R}_s \times (L \cap \text{ff})$ and away from $\mathbb{R}_s \times R$, or near $\mathbb{R}_s \times (R \cap \text{ff})$ and away from $\mathbb{R}_s \times L$.

Region 3: Near $\mathbb{R}_s \times (L \cap R)$ but away from $\mathbb{R}_s \times \text{ff}$.

Region 4: Near $\mathbb{R}_s \times (L \cap R \cap \text{ff})$.

First we analyze region 1, near $\mathbb{R}_s \times L$ but away from $\mathbb{R}_s \times R$ and $\mathbb{R}_s \times \text{ff}$. The case near $\mathbb{R}_s \times R$ but away from $\mathbb{R}_s \times L$ and $\mathbb{R}_s \times \text{ff}$ is identical. Since we are away from R , we have $\rho_R > \delta$, for some $\delta > 0$, and hence $\log \rho_R$ is C^∞ . In this region we may take x as a defining function of L , and instead of (1.5), we set $s = t + \log x$. In fact, the map $(s, m) \mapsto (s + \log \rho_R, m)$ is a diffeomorphism in the region where $\rho_R > \delta$, and hence the statements about q_L and H_{q_L} in the lemma are true in this region whether we take $s = t + \log x$ or $s = t + \log x + \log \rho_R$. In the case near $\mathbb{R}_s \times R$ but away from $\mathbb{R}_s \times L$ and $\mathbb{R}_s \times \text{ff}$ one sets $s = t + \log x'$. These particular cases were studied in [12].

The change of variables

$$(2.7) \quad s = t + \log x$$

induces the symplectic change on $T^*(\mathbb{R} \times \mathring{X} \times \mathring{X})$

$$(2.8) \quad \begin{aligned} (x, y, \xi, \eta, t, \tau) &\mapsto (x, y, \tilde{\xi}, \eta, s, \sigma), \\ \text{where } \tilde{\xi} &= \xi - \frac{1}{x}\tau, \quad \sigma = \tau. \end{aligned}$$

In coordinates (1.3),

$$g_L^*(x, y, \xi, \eta) = x^2 \xi^2 + x^2 h(x, y, \eta),$$

and so

$$\begin{aligned} \beta_1^* Q_L &= -x\sigma\tilde{\xi} - \frac{1}{2}x^2(\tilde{\xi}^2 + h(x, y, \eta)), \text{ and hence} \\ q_L &= -\frac{1}{\rho_L}\beta_1^* Q_L = \sigma\tilde{\xi} + \frac{1}{2}x(\tilde{\xi}^2 + h(x, y, \eta)). \end{aligned}$$

We have

$$H_{q_L} = (\sigma + x\tilde{\xi})\partial_x + \tilde{\xi}\partial_s + \frac{1}{2}xH_{h(x, y, \eta)} - \frac{1}{2}(\tilde{\xi}^2 + h(x, y, \eta) + x\partial_x h(x, y, \eta))\partial_{\tilde{\xi}}.$$

In particular, σ remains constant along the integral curves of H_{q_R} , and

$$H_{q_L}|_{\{x=0\}} = \sigma\partial_x + \tilde{\xi}\partial_s - \frac{1}{2}(\tilde{\xi}^2 + h(0, y, \eta))\partial_{\tilde{\xi}}.$$

So if $\sigma \neq 0$, H_{q_L} is transversal to ∂X .

Next we work in region 2 near $\mathbb{R}_s \times (L \cap \text{ff})$, but away from $\mathbb{R}_s \times R$. The case near $\mathbb{R}_s \times (R \cap \text{ff})$ but away from $\mathbb{R}_s \times L$ is very similar. In this case, $\rho_R = x'/R > \delta$, and so it is better to use projective coordinates

$$(2.9) \quad X = \frac{x}{x'}, \quad Y = \frac{y - y'}{x'}, \quad x' \text{ and } y'.$$

In this case, X is a boundary defining function for L and x' is a boundary defining function for ff . Since β_0 is a diffeomorphism in the interior of $X \times_0 X$, it induces a symplectic change of variables

$$(x, y, \xi, \eta, x', y', \xi', \eta') \in T^*(\mathring{X} \times \mathring{X}) \mapsto (X, Y, \lambda, \mu, x', y', \lambda', \mu') \in T^*(X \times_0 X),$$

given by

$$\lambda = x'\xi, \quad \mu = x'\eta, \quad \lambda' = \xi' + \xi X + \eta Y \text{ and } \mu' = \eta + \eta'.$$

and Q_L becomes

$$\beta_0^* Q_L = \frac{1}{2}(\tau^2 - X^2(\lambda^2 + h(x'X, x'Y + y', \mu))),$$

and here we used the fact that $h(x, y, \eta)$ is homogeneous of degree two in η .

Away from the face R , $\rho_R > \delta$, for some δ , and the function $\log \rho_R$ is smooth. Therefore, as argued above in the case of region 1, the transformation $(s, m) \mapsto (s + \log \rho_R, m)$ is a C^∞ map away from $\{\rho_R = 0\}$, and so it suffices to take

$$(2.10) \quad s = t + \log X.$$

The change of variable (2.10) induces the following symplectic change of variables

$$(2.11) \quad \begin{aligned} T^*(\mathbb{R}_t \times \dot{X} \times \dot{X}) &\longrightarrow T^*(\mathbb{R}_s \times X \times_0 X), \\ (t, \tau, x, y, \xi, \eta, x', y', \xi', \eta') &\longmapsto (s, \sigma, X, Y, \tilde{\lambda}, \mu, x', y', \lambda', \mu') \\ \text{where } \tilde{\lambda} &= \lambda - \frac{\tau}{X}, \quad \sigma = \tau, \end{aligned}$$

and the canonical 2-form on $T^*(\mathbb{R}_s \times X \times_0 X)$ is given by

$$\omega^0 = d\tilde{\lambda} \wedge dX + d\mu \wedge dY + d\lambda' \wedge dx' + d\mu' \wedge dy'.$$

Hence

$$\beta_1^* Q_L = -\tilde{\lambda}\sigma X - \frac{1}{2}X^2(\tilde{\lambda}^2 + h(x'X, x'Y + y', \mu)),$$

and we conclude that

$$q_L = -\frac{1}{\rho_L}\beta_1^* Q_L = -\frac{1}{X}\beta_1^* Q_L = \tilde{\lambda}\sigma + \frac{1}{2}X(\tilde{\lambda}^2 + h(x'X, x'Y + y', \mu)).$$

Hence vector field H_{q_L} is given by

$$(2.12) \quad H_{q_L} = \tilde{\lambda} \frac{\partial}{\partial s} + (\sigma + X\tilde{\lambda}) \frac{\partial}{\partial X} - \frac{1}{2}(\tilde{\lambda}^2 + h + x'X\partial_X h) \frac{\partial}{\partial \tilde{\lambda}} + \frac{X}{2}H_h + T,$$

where T is a smooth vector field in $\partial_X, \partial_{\mu'}$. So away from $\sigma = 0$, H_{q_L} is transversal to $\mathbb{R}_s \times L$.

Next we analyze region 3, near $\mathbb{R}_s \times (L \cap R)$ and away from $\mathbb{R}_s \times \text{ff}$. Here x, x' are boundary defining functions for $\mathbb{R}_s \times L$ and $\mathbb{R}_s \times R$ respectively. In this case, as discussed above, we can take

$$s = t + \log x + \log x',$$

which induces the following symplectic change of variable

$$\begin{aligned} (t, \tau, x, y, \xi, \eta, x', y', \xi', \eta') &\longmapsto (s, \sigma, x, y, \tilde{\xi}, \eta, x', y', \tilde{\xi}', \eta'), \\ \text{where } \tilde{\xi} &= \xi - \frac{\tau}{x}, \quad \tilde{\xi}' = \xi' - \frac{\tau}{x'}, \quad \sigma = \tau. \end{aligned}$$

The symbols can be computed as in the case near $\mathbb{R}_s \times L$ away from $\mathbb{R}_s \times \text{ff}$ and $\mathbb{R}_s \times R$. In particular,

$$\begin{aligned} \beta_1^* Q_L &= -x\sigma\tilde{\xi} - \frac{1}{2}x^2(\tilde{\xi}^2 + h(x, y, \eta)) \text{ and so } q_L = -\frac{1}{\rho_L}\beta_1^* Q_L = \sigma\tilde{\xi} + \frac{1}{2}x(\tilde{\xi}^2 + h(x, y, \eta)), \\ \beta_1^* Q_R &= -x\sigma\tilde{\xi}' - \frac{1}{2}x'^2(\tilde{\xi}'^2 + h(x', y', \eta')) \text{ and so } q_R = \sigma\tilde{\xi}' + \frac{1}{2}x'(\tilde{\xi}'^2 + h(x', y', \eta')). \end{aligned}$$

The Hamilton vector fields are given by

$$\begin{aligned} H_{q_L} &= (\sigma + x\tilde{\xi})\partial_x + \tilde{\xi}\partial_s + \frac{1}{2}xH_{h(x,y,\eta)} - \frac{1}{2}(\tilde{\xi}^2 + h(x,y,\eta) + x\partial_x h(x,y,\eta))\partial_{\tilde{\xi}}, \\ H_{q_R} &= (\sigma + x'\tilde{\xi}')\partial_{x'} + \tilde{\xi}'\partial_s + \frac{1}{2}x'H_{h(x',y',\eta')} - \frac{1}{2}(\tilde{\xi}'^2 + h(x',y',\eta') + x'\partial_{x'} h(x',y',\eta'))\partial_{\tilde{\xi}'}. \end{aligned}$$

We conclude that, away from $\sigma = 0$, H_{q_L} is transversal to $\mathbb{R}_s \times L = \{x = 0\}$ while H_{q_R} is transversal to $\mathbb{R}_s \times R = \{x' = 0\}$.

Finally, we analyze region 4, near the co-dimension 3 corner $\mathbb{R}_s \times (L \cap \text{ff} \cap R)$. Here we also work with suitable projective coordinates, and without loss of generality, as in [10] we may take $\rho_{\text{ff}} = y_1 - y'_1 \geq 0$ and take the following coordinates

$$(2.13) \quad u = y_1 - y'_1, \quad w = \frac{x}{y_1 - y'_1}, \quad w' = \frac{x'}{y_1 - y'_1}, \quad y' \text{ and } Z_j = \frac{y_j - y'_j}{y_1 - y'_1}, \quad j = 2, 3, \dots, n.$$

Here w, w' and u are boundary defining functions for $\mathbb{R}_s \times L$, $\mathbb{R}_s \times R$ and $\mathbb{R}_s \times \text{ff}$ faces respectively. The induced symplectic change of variables

$$\begin{aligned} T^*(\mathring{X} \times \mathring{X}) &\longrightarrow T^*(X \times_0 X) \\ (x, y, \xi, \eta, x', y', \xi', \eta') &\longmapsto (w, u, Z, \lambda, \nu, \mu, w', y', \lambda', \mu') \\ &\text{where} \\ (2.14) \quad \lambda &= \xi u, \quad \lambda' = \xi' u, \quad \nu = \xi w + \xi' w' + \eta_1 + \sum_{j=2}^n \eta_j Z_j, \\ \mu' &= \eta + \eta', \quad \mu_j = \eta_j u, \quad j = 2, 3, \dots, n. \end{aligned}$$

In these coordinates, the symbols of Q_L and Q_R are given by

$$\begin{aligned} \beta_0^* Q_L &= \frac{1}{2}(\tau^2 - w^2(\lambda^2 + h(uw, y, u\eta))), \\ \beta_0^* Q_R &= \frac{1}{2}(\tau^2 - w'^2(\lambda'^2 + h(uw', y', u\mu' - u\eta))), \end{aligned}$$

where

$$y = (y'_1 + u, y'_2 + uZ_2, \dots, y'_n + uZ_n), \quad u\eta = (u\nu - \lambda w - \lambda' w' - \sum_{j=2}^n \mu_j Z_j, \mu).$$

In this case, we set

$$(2.15) \quad s = t + \log w + \log w',$$

which induces the symplectic transformation

$$\begin{aligned} T^*(\mathbb{R}_t \times \mathring{X} \times \mathring{X}) &\longrightarrow T^*(\mathbb{R}_s \times X \times_0 X), \\ (t, \tau, x, y, \xi, \eta, x', y', \xi', \eta') &\longmapsto (s, \sigma, w, u, Z, \tilde{\lambda}, \nu, \mu, w', y', \tilde{\lambda}', \mu') \\ \text{where } \tilde{\lambda} &= \lambda - \frac{\tau}{w}, \quad \tilde{\lambda}' = \lambda' - \frac{\tau}{w'}, \quad \sigma = \tau. \end{aligned}$$

Here the canonical 2-form on $T^*(\mathbb{R}_s \times X \times_0 X)$ is given by

$$\omega^0 = d\sigma \wedge ds + d\tilde{\lambda} \wedge dw + d\tilde{\lambda}' \wedge dw' + d\nu \wedge du + d\mu \wedge dZ + d\mu' \wedge dy'.$$

The lift of the symbols Q_L and Q_R become

$$\beta_1^* Q_L = -w\sigma\tilde{\lambda} - \frac{1}{2}w^2(\tilde{\lambda}^2 + h(uw, y, \tilde{\eta})),$$

$$\beta_1^* Q_R = -w'\sigma\tilde{\lambda}' - \frac{1}{2}w'^2(\tilde{\lambda}'^2 + h(uw', y', u\mu' - \tilde{\eta})),$$

where $\tilde{\eta} \doteq u\eta = (u\nu - \tilde{\lambda}w - \tilde{\lambda}'w' - 2\sigma - \sum_{j=2}^n \mu_j Z_j, \mu)$. Therefore, in these coordinates

$$q_L = -\frac{1}{\rho_L}\beta_1^* Q_L = \sigma\tilde{\lambda} + \frac{1}{2}w(\tilde{\lambda}^2 + h(uw, y, \tilde{\eta})),$$

$$q_R = -\frac{1}{\rho_R}\beta_1^* Q_R = \sigma\tilde{\lambda}' + \frac{1}{2}w'(\tilde{\lambda}'^2 + h(uw', y', u\mu' - \tilde{\eta})).$$

Hence the Hamilton vector fields are of the form

$$(2.16) \quad \begin{aligned} H_{q_L} &= (\sigma + w\tilde{\lambda} - \frac{1}{2}w^2\partial_{\tilde{\eta}}h(uw, y, \tilde{\eta}))\frac{\partial}{\partial w} + T_L, \\ H_{q_R} &= (\sigma + w'\tilde{\lambda}' + \frac{1}{2}w'^2\partial_{\tilde{\eta}}h(uw', y', u\mu' - \tilde{\eta}))\frac{\partial}{\partial w'} + T_R, \end{aligned}$$

where T_L, T_R are smooth vector fields on $T^*(\mathbb{R}_s \times X \times_0 X)$ with no $\frac{\partial}{\partial w}, \frac{\partial}{\partial w'}$ or $\frac{\partial}{\partial \sigma}$ terms. Notice that these vector fields are C^∞ up to the front face, and that away from $\sigma = 0$, the vector H_{q_L} is transversal to $\mathbb{R}_s \times L$ and H_{q_R} is transversal to $\mathbb{R}_s \times R$. This shows that the transversality to L and R holds up to the corner. This ends the proof of the Lemma. \square

Now we conclude the proof of Theorem 1.1.

Proof. Since in the interior of $X \times_0 X$, β_0 is a C^∞ diffeomorphism between C^∞ open manifolds, $\beta_0^* \tilde{\Lambda}$ is a C^∞ Lagrangian manifold in the interior of $\mathbb{R}_t \times X \times_0 X$, and it is defined as

$$\beta_0^* \tilde{\Lambda} = \bigcup_{t_1 \geq 0, t_2 \geq 0} \exp t_2 \beta_0^* H_{Q_L} \circ \exp t_1 \beta_0^* H_{Q_R} (\beta_0^* \Sigma),$$

where

$$\Sigma = \{(0, \tau, x, y, \xi, \eta, x, y, -\xi, -\eta) : x^2 \xi^2 + x^2 h(x, y, \eta) = \tau^2\}.$$

In projective coordinates

$$x', \quad X = \frac{x}{x'}, \quad Y = \frac{y - y'}{x'}, \quad y',$$

valid near ff and L , $\beta_0^* \Sigma$ can be written as

$$\beta_0^* \Sigma = \{(0, 1, X, Y, \lambda, \mu, x', y', \lambda', \mu') : X = 1, Y = 0, \lambda' = \mu' = 0, \lambda^2 + h(x', y', \mu) = \tau^2\},$$

which is a C^∞ submanifold of $T^*(\mathbb{R}_t \times X \times_0 X)$ that extends smoothly up to the front face $\mathbb{R}_t \times \text{ff} = \{x' = 0\}$. Since $\beta_0^* \Sigma$ does not intersect either $\mathbb{R}_s \times L$ or $\mathbb{R}_s \times R$, these properties do not change if we set $s = t + \log \rho_R + \log \rho_L$, and hence $\beta_1^* \Sigma$ is a C^∞ submanifold of $T^*(\mathbb{R}_s \times X \times_0 X)$ that has a C^∞ extension up to $\mathbb{R}_s \times \text{ff}$.

In the interior of $\mathbb{R}_s \times X \times_0 X$, $\beta_1^* Q_L$ and $\beta_1^* Q_R$ vanish on $\beta_1^* \tilde{\Lambda}$, and hence the integral curves of H_{q_L} and H_{q_R} on $\beta_1^* \tilde{\Lambda}$ coincide with the integral curves of $H_{\beta_1^* Q_L}$ and $H_{\beta_1^* Q_R}$ respectively. Therefore, in the interior of $\mathbb{R}_s \times X \times_0 X$ and across to the front face, $\beta_1^* \tilde{\Lambda}$ is the union of integral curves of H_{q_L} and H_{q_R} emanating from $\beta_1^* \Sigma$.

Since Q_R and Q_L do not depend on t , it follows that q_L and q_R do not depend on s , and hence σ remains constant along the integral curves of q_L and q_R . Since $\sigma = \tau \neq 0$ on $\beta_1^* \Sigma$, it follows that

$\sigma \neq 0$ on $\beta_1^* \tilde{\Lambda}$ in the interior of $\mathbb{R}_s \times X \times_0 X$. However, we have also shown that, up to the front face, in the region $\sigma = 1$, H_{q_L} is transversal to $\mathbb{R}_s \times L$ while H_{q_R} is transversal up to $\mathbb{R}_s \times R$.

Recall from (2.12) that H_{q_L} and H_{q_R} are C^∞ up to $\mathbb{R}_s \times \text{ff}$ and are tangent to $\mathbb{R}_s \times \text{ff}$. So, $\beta_1^* \tilde{\Lambda}$ extends up to $\mathbb{R}_s \times \text{ff}$ as the joint flow-out of $\beta_1^* \Sigma$ by H_{q_R} and H_{q_L} .

So the integral curves of H_{q_L} can be continued smoothly up to $\mathbb{R}_s \times L$ and the integral curves of H_{q_R} can be continued smoothly up to $\mathbb{R}_s \times R$. Therefore $\beta_1^* \tilde{\Lambda}$ can be extended up to the face $\{\rho_R = 0\}$ because H_{q_R} is tangent to $\beta_1^* \tilde{\Lambda}$ and transversal to $\{\rho_R = 0\}$. The same holds for the left face. This shows that $\beta_1^* \tilde{\Lambda}$, which is in principle is defined in the interior of $\mathbb{R}_s \times X \times_0 X$, extends to a C^∞ manifold up to $\partial(\mathbb{R}_s \times X \times_0 X)$ which intersects $\mathbb{R}_s \times L$ and $\mathbb{R}_s \times R$ transversally. See Fig. 4.

We can make this more precise if we work suitable local symplectic coordinates valid near a point on the fiber over the corner $\text{ff} \cap L \cap R$. We know that R , L and ff intersect transversally. So one can choose local coordinates $x = (x_1, x_2, x_3, x')$ in \mathbb{R}^{2n+2} valid near $\text{ff} \cap L \cap R$ such that

$$\text{ff} = \{x_3 = 0\}, \quad R = \{x_1 = 0\} \text{ and } L = \{x_2 = 0\}.$$

and that the symplectic form $\omega^0 = d\sigma \wedge ds + d\xi \wedge dx$. For example, this can be accomplished by using local coordinates defined in (2.13) and setting $u = x_3$, $w = x_2$ and $w' = x_1$, $(y', Z) = x'$.

We know that $\beta_0^* \tilde{\Lambda}$ is a Lagrangian submanifold of $T^*(\mathbb{R}_s \times \mathbb{R}^{2n+2})$ contained in $\{x_1 > 0, x_2 > 0, x_3 \geq 0\}$, which intersects $\text{ff} = \{x_3 = 0\}$ transversally. There are commuting Hamilton vector fields H_{q_R} and H_{q_L} tangent to $\beta_0^* \tilde{\Lambda}$ that are C^∞ up to $\{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\}$, and as long as $\sigma \neq 0$, H_{q_R} transversal to R and tangent to L and ff and H_{q_L} is transversal to L and tangent to R and ff . Also, since q_R and q_L do not depend on s , σ remains constant along the integral curves of H_{q_R} and H_{q_L} .

Let

$$\mathcal{F} = T_{\{x_1=x_2=0\}}^*(\mathbb{R}_s \times \{x : x_1 > 0, x_2 > 0, x_3 \geq 0\}),$$

and let $p = (s, \sigma, 0, \xi_1, 0, \xi_2, x_3, \xi_3, x', \xi')$, $\sigma \neq 0$, denote a point on \mathcal{F} , where

Since q_R and q_L do not depend on s , σ remains constant along the integral curves of H_{q_R} and H_{q_L} . Moreover, in the region $\sigma \neq 0$, the vector fields H_{q_R} and H_{q_L} are smooth, nondegenerate up to the boundaries. H_{q_R} is tangent to ff and L , while H_{q_L} is tangent to ff and R . So, for ε small enough we define

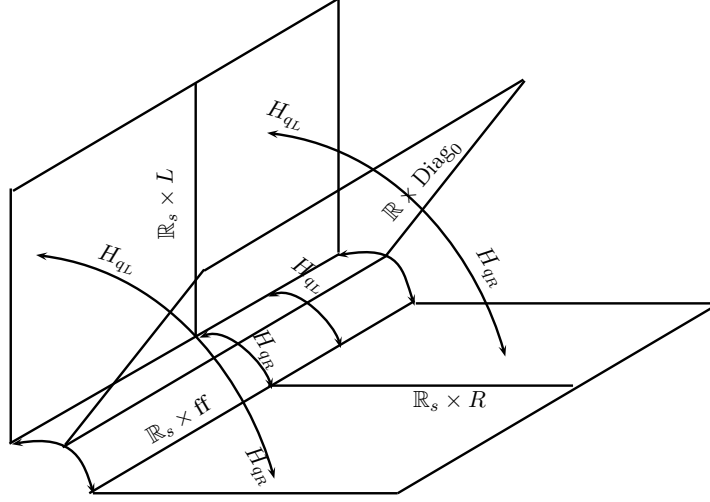
$$\begin{aligned} \Psi_0 : [0, \varepsilon) \times [0, \varepsilon) \times (\mathcal{F} \cap \{\sigma \neq 0\}) &\longrightarrow U_0 \subset T^*(\mathbb{R}_s \times \{x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}) \\ \Psi_0(t_1, t_2, p) &= \exp(-t_1 H_{q_R}) \circ \exp(-t_2 H_{q_L})(p), \end{aligned}$$

and

$$\begin{aligned} \Psi_1 : [0, \varepsilon) \times [0, \varepsilon) \times (\mathcal{F} \cup \{\sigma \neq 0\}) &\longrightarrow U_1 \subset T^*(\mathbb{R}_s \times \{x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}) \\ \Psi_1(t_1, t_2, p) &= \exp(-t_1 \partial_{x_1}) \circ \exp(-t_2 \partial_{x_2})(p), \end{aligned}$$

Since the vector fields H_{q_R} , H_{q_L} commute and ∂_{x_1} and ∂_{x_2} commute, both maps are C^∞ map and moreover,

$$\begin{aligned} \Psi_0^* H_{q_R} &= -\partial_{t_1}, & \Psi_0^* H_{q_L} &= -\partial_{t_2} \\ \Psi_1^* \partial_{x_1} &= -\partial_{t_1}, & \Psi_1^* \partial_{x_2} &= -\partial_{t_2} \end{aligned}$$

FIGURE 4. The integral curves of H_{q_R} and H_{q_L} in the region $\sigma = 1$.

Hence,

$$\begin{aligned}\Psi &= \Psi_0 \circ \Psi_1^{-1} : U_1 \longrightarrow U_0, \\ \Psi^* H_{q_R} &= -\partial_{x_1}, \quad \Psi^* H_{q_L} = -\partial_{x_2}.\end{aligned}$$

Moreover, if ω^0 is the symplectic form on $T^*(\mathbb{R} \times X \times_0 X)$, in coordinates (2.13) valid near \mathcal{F} ,

$$\Psi^* \omega^0 = \omega^0.$$

Now $\Upsilon = \Psi^{-1}(\beta_0^* \tilde{\Lambda})$ is a C^∞ Lagrangian in $\{x_1 > 0, x_2 > 0, x_3 \geq 0\}$ which intersects $\{x_3 = 0\}$ transversally, and both ∂_{x_1} and ∂_{x_2} are tangent to Υ . But this implies that for any point $p \in \Upsilon$, the integral curves of ∂_{x_j} , $j = 1, 2$ starting at a point $p \in \Upsilon$ are contained in Υ . Therefore, for any $p = (x_1, \xi_1, x_2, \xi_2, x_3, \xi_3, x', \xi') \in \Upsilon$, with x_1 and x_2 small enough, the set $\{x_1 - t_1, \xi_1, x_2 - t_2, \xi_2, x_3, \xi_3, x', \xi'\} \subset \Upsilon$. By taking t_1 and t_2 large enough, this gives an extension $\overline{\Upsilon}$ of Υ to $\{x_1 \leq 0\} \cup \{x_2 \leq 0\}$. Now $\Psi(\overline{\Upsilon})$ is the desired Lagrangian extension of $\beta_0^* \tilde{\Lambda}$. Notice that in fact, it extends past the boundaries $\{x_1 = 0\}$ and $\{x_2 = 0\}$. The construction in the other regions, away from the co-dimension three corners follows by the same argument. \square

We remark that if Λ_1 is the extension of $\beta_1^* \tilde{\Lambda}$, then, as mentioned in the introduction, the scattering relation is defined as the intersection of Λ_1 to the corner $\mathbb{R}_s \times \{\rho_R = \rho_L = 0\}$. Namely, $SR = \Lambda_1 \cap T^*_{(\mathbb{R}_s \times \{\rho_R = \rho_L = 0\})}(\mathbb{R}_s \times X \times_0 X)$. Notice that the manifold $\{\rho_R = \rho_L = 0\} = \partial X \times_0 \partial X$ is the blow-up of $\partial X \times \partial X$ induced by β_0 , and we let $\beta_{0, \partial X}$ denote the blow-down map introduced in [7].

It is worth observing that Theorem 1.1 extends the result of [12] to the case of where $(z, z') \rightarrow q \in \partial X \times \partial X$. Let $(z_0, \zeta_0, z_0, -\zeta_0)$, $|\zeta_0|_{g^*(z_0)} = 1$, and let

$$(z(t), \zeta(t), z'(t), \zeta'(t)) = \exp\left(\frac{t}{2} H_{p_R}\right) \exp\left(\frac{t}{2} H_{p_L}\right) (z_0, \zeta_0, z_0, -\zeta_0), \quad t > 0.$$

Let γ be the curve defined by this equation joining (z, ζ) and (z', ζ') . Suppose that $\beta_0^{-1}(z(t), z'(t)) \rightarrow m \in \partial X \times_0 \partial X$ as $t \rightarrow \infty$. The variable s is a C^∞ function on the Lagrangian $\beta_0^*(\tilde{\Lambda})$ up to $\partial(T^*(\mathbb{R}_s \times X \times_0 X))$. In particular $s(\beta_0^*(z, \zeta, z', \zeta'))$ is a C^∞ function of the initial data (z_0, ζ_0) , with $|\zeta_0| = 1$, up to the boundary. In particular, its restriction to the boundary is a C^∞ function, and we have

$$(2.17) \quad S(m, z_0, \zeta_0) = \lim_{t \rightarrow \infty} s(\beta_0^*((z(t), \zeta(t), z'(t), \zeta'(t)))) \in C^\infty(\partial X \times_0 \partial X \times S^*\tilde{X}).$$

In particular if there exist open subsets $U, V \subset \partial X$ such that $z(t) \rightarrow y \in U$, $z'(t) \rightarrow y' \in V$, and $U \cap V = \emptyset$, then $S(y, y', z_0, \zeta_0) \in C^\infty(U \times V \times S^*\tilde{X})$.

The definition of the sojourn time along a geodesic from [12] can be extended to the case where both points approach ∂X as

$$S_{soj} = \beta_0^*(t + \log x + \log x') = s + 2 \log R,$$

and in particular $S_{soj}|_{\{\rho_R=\rho_L=0\}} = s|_{\{\rho_R=\rho_L=0\}} + 2 \log R|_{\{\rho_R=\rho_L=0\}}$, and hence

$$(2.18) \quad \beta_0^* e^{-i\lambda S_{soj}} \Big|_{\{\rho_R=\rho_L=0\}} = R^{-2i\lambda} H|_{\{\rho_R=\rho_L=0\}}, \quad H \in C^\infty(X \times_0 X)$$

which according to [7] is a multiple the most singular term in the expansion of the Schwartz kernel of the scattering matrix.

3. ASYMPTOTICS OF THE DISTANCE FUNCTION

Let

$$(3.1) \quad p(z, \zeta) = \frac{1}{2}(|\zeta|_{g^*(z)}^2 - 1), \quad \text{where } |\zeta|_{g^*(z)}^2 = g(z)^*(\zeta, \zeta),$$

and let

$$(3.2) \quad S^*\text{Diag} = \{(z, \zeta, z', \zeta') \in T^*(\tilde{X} \times \tilde{X}) : z = z', \zeta = \zeta'; p(z', \zeta') = 0\}.$$

We define the Lagrangian submanifold of $T^*(\tilde{X} \times \tilde{X})$,

$$(3.3) \quad \Lambda = \bigcup_{t \geq 0} \exp(tH_p) S^*\text{Diag} = \{(z, \zeta, z', \zeta') : \exists t \geq 0, (z, \zeta) = \exp(tH_p)(z', \zeta'), p(z', \zeta') = 0\}$$

It is well-known that integral curves of H_p contained in $\{p = 0\}$ project onto geodesics of the metric g , see section 2.7 of [1]. In other words, if $(z', \zeta') \in T^*\tilde{X}$ and $p(z', \zeta') = 0$, and if $\gamma(r)$ is a curve such that

$$\begin{aligned} \frac{d}{dr} \gamma(r) &= H_p(\gamma(r)), \\ \gamma(0) &= (z', -\zeta'). \end{aligned}$$

Then, with \mathcal{G} given by (2.1),

$$(3.4) \quad \begin{aligned} \gamma(r) &= \left(\alpha(r), \mathcal{G}\left(\frac{d}{dr} \alpha(r)\right) \right), \quad \text{where} \\ \alpha(r) &= \exp_{z'}(rv), \quad v \in T_{z'}\tilde{X}, \quad |v|_g^2 = 1, \quad \mathcal{G}(v) = -\zeta', \end{aligned}$$

$\exp_{z'}(r\bullet)$ denotes the exponential map on $T_{z'}\tilde{X}$. Now we assume that (\tilde{X}, g) is geodesically convex. In this case, the exponential map $\exp_{z'}(r\bullet)$ is a global diffeomorphism for all $z' \in \tilde{X}$. If

$$S_{z'} = \{v \in T_{z'}\tilde{X} : |v|_g^2 = 1\},$$

the exponential map $\exp_{z'}$ gives geodesic normal coordinates about the point z' :

$$\begin{aligned} S_{z'} \times [0, \infty) &\longrightarrow \mathring{X} \\ (v, r) &\mapsto \exp_{z'}(rv). \end{aligned}$$

The geodesic is given by $\exp_{z'}(rv) = (z', v, r)$, and its tangent vector is ∂_r . Moreover, the metric g takes the form

$$g = dr^2 + r^2 H(z, r, v, v),$$

and in this case $\mathcal{G}(\partial_r) = dr$, and according to (3.4), the integral curve of H_p starting at $(z', -\zeta')$ is given by

$$(3.5) \quad \gamma(r) = (z', -\zeta', v, r, dr).$$

So, the starting point of the curve $\gamma(r)$ is $(z', -\zeta')$ and its end point is (z, ζ) with $\zeta = dr$. If one now reverses the role of z and z' , and follows the geodesic in the opposite orientation, the starting point is $(z, -\zeta)$, and the end point is (z', ζ') , with $\zeta' = dr$. Then (3.3) and (3.5) give that

Proposition 3.1. *Let (\mathring{X}, g) be a geodesically convex AHM. Then the manifold Λ defined in (3.3) is a C^∞ embedded Lagrangian submanifold, and away from the diagonal Λ is the graph of the differential of the distance function. In other words,*

$$(3.6) \quad \Lambda \setminus S^* \text{Diag} = \{(z, \zeta, z', \zeta') : \zeta' = -d_{z'}r(z, z'), \zeta = d_z r(z, z'), \text{ provided } (z, \zeta) \neq (z', -\zeta')\}.$$

Proof of Theorem 1.2. According to Proposition 3.1, away from the diagonal, the Lagrangian $\tilde{\Lambda}$ defined in (2.5) satisfies

$$\tilde{\Lambda} \cap \{\tau = 1\} = \{(t, 1, z, \zeta, z', \zeta') \in T^*(\mathbb{R}_t \times \mathring{X} \times \mathring{X}) : (z, \zeta) = \exp(tH_p)(z', \zeta'), p(z', \zeta') = 0\},$$

On the other hand, in view of (3.6), away from the diagonal,

$$\tilde{\Lambda} \cap \{\tau = 1\} = \{(t, \tau, z, \zeta, z', \zeta') \in T^*(\mathbb{R}_t \times \mathring{X} \times \mathring{X}) : \tau = 1, t = r(z, z'), \zeta = d_z r(z, z'), \zeta' = d_{z'} r(z, z')\}.$$

Theorem 1.1, guarantees that manifold $\tilde{\Lambda} \cap \{\tau = 1\}$ has a smooth extension to $T^*(\mathbb{R}_s \times X \times_0 X)$, where away from Diag_0 ,

$$(3.7) \quad s = t + \log \rho_L + \log \rho_R = r(z, z') + \log \rho_R + \log \rho_L.$$

In particular, the right hand side of (3.7) shows that s is in fact a function of the base variables only, and $s \in C^\infty(X \times_0 X \setminus \text{Diag}_0)$. \square

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